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Neutral Minima in Three-Higgs Doublet Models

A. Tofighi · M. Moazzen

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Abstract We study the structure of the most general three Higgs doublet model. By imposing discrete and continuous symmetry on the scalar fields the number of parameters of the model is drastically reduced. It is possible to describe the model in terms of nine invariants. We use the formalism to study an specific model with eight real parameters. We discuss the structure of neutral minima for this model.

Keywords Three-Higgs doublet model · Neutral minima

1 Introduction

The standard model of electroweak interactions has been successful in describing currently known experimental data [1]. But the scalar sector of the theory has not been directly tested so far. With the advent of Large Hadron Collider (*LHC*) the expectation is that we test this sector of theory as well.

In the standard model the scalar sector is comprised of a $SU(2)$ Higgs doublet. But if nature is supersymmetric the scalar sector must contain two such doublets [2]. In addition two Higgs doublet models provide solutions for problems such as baryogenesis and CP violation [3–7].

Various extensions of the scalar sector has been the subject of numerous studies in recent years [8–11]. As noted in [11] it is easy to construct models for N Higgs doublet case. But the problem is that the number of parameters of the models grows very rapidly with N. So it is difficult to make a qualitative study of the Higgs sector for $N > 2$. In Ref. [12] the vacuum structure of general multi-Higgs doublet model has been investigated. But this study is of a qualitative nature.

Our motivation is to make a quantitative discussion for the case of $N = 3$. The organization of this paper as follows:

A. Tofighi (✉) · M. Moazzen
Department of Physics, Faculty of Basic Science, University of Mazandaran, P.O. Box 47416-1467, Babolsar, Iran
e-mail: A.Tofighi@umz.ac.ir

In section two we generalize the method of Ref. [7] to the case of three Higgs doublet. We provide expressions for the nine $SU(2) \times U(1)$ real quadratic invariants of the model. A general three Higgs doublet potential has 54 real parameters. We show that by imposing multiple symmetries such as global phase invariance one can reduce this number. In section three we consider a specific $3HDM$ model. This model has only eight real parameters. We study the vacuum structure of this model for the case of neutral minima. In section four we obtain the general form of the mass matrices for the Higgs sector. By studying the positivity of the mass matrices, we obtain some conditions on the coefficients of the model. In section five we calculate the value of the potential at these minima. In section six we discuss the case of CP breaking stationary points. And finally in section seven we present our conclusions.

2 The Three-Higgs Doublet Model Potential

In $3HDM$ one utilizes three complex scalar doublets, defined by

$$\Phi_1 = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_7 + i\varphi_{10} \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} \varphi_3 + i\varphi_4 \\ \varphi_8 + i\varphi_{11} \end{pmatrix}, \qquad \Phi_3 = \begin{pmatrix} \varphi_5 + i\varphi_6 \\ \varphi_9 + i\varphi_{12} \end{pmatrix}, \tag{1}$$

with hypercharge assignments

$$Y(\Phi_1) = Y(\Phi_2) = Y(\Phi_3) = 1. \tag{2}$$

The fields φ_i are all real functions. Now in general with N Higgs doublet it is possible to build N^2 real quadratic forms that are invariant under $SU(2) \times U(1)$. Four our case these invariants are

$$x_1 \equiv \Phi_1^\dagger \Phi_1 = \varphi_1^2 + \varphi_2^2 + \varphi_7^2 + \varphi_{10}^2 \tag{3}$$

$$x_2 \equiv \Phi_2^\dagger \Phi_2 = \varphi_3^2 + \varphi_4^2 + \varphi_8^2 + \varphi_{11}^2 \tag{4}$$

$$x_3 \equiv \Phi_3^\dagger \Phi_3 = \varphi_5^2 + \varphi_6^2 + \varphi_9^2 + \varphi_{12}^2 \tag{5}$$

$$x_4 \equiv Re(\Phi_1^\dagger \Phi_2) = \varphi_1\varphi_3 + \varphi_2\varphi_4 + \varphi_7\varphi_8 + \varphi_{10}\varphi_{11} \tag{6}$$

$$x_5 \equiv Re(\Phi_1^\dagger \Phi_3) = \varphi_1\varphi_5 + \varphi_2\varphi_6 + \varphi_7\varphi_9 + \varphi_{10}\varphi_{12} \tag{7}$$

$$x_6 \equiv Re(\Phi_2^\dagger \Phi_3) = \varphi_3\varphi_5 + \varphi_4\varphi_6 + \varphi_8\varphi_9 + \varphi_{11}\varphi_{12} \tag{8}$$

$$x_7 \equiv Im(\Phi_1^\dagger \Phi_2) = \varphi_1\varphi_4 - \varphi_2\varphi_3 + \varphi_7\varphi_{11} - \varphi_8\varphi_{10} \tag{9}$$

$$x_8 \equiv Im(\Phi_1^\dagger \Phi_3) = \varphi_1\varphi_6 - \varphi_2\varphi_5 + \varphi_7\varphi_{12} - \varphi_9\varphi_{10} \tag{10}$$

$$x_9 \equiv Im(\Phi_2^\dagger \Phi_3) = \varphi_3\varphi_6 - \varphi_4\varphi_5 + \varphi_8\varphi_{12} - \varphi_9\varphi_{11}. \tag{11}$$

The most general $3HDM$potential [ ] is

$$V_g(\Phi_1, \Phi_2, \Phi_3) = \sum_{i=1}^{9} a_i x_i + \frac{1}{2} \sum_{j=1}^{9} \sum_{i=1}^{9} b_{ij} x_i x_j, \tag{12}$$

where the coefficients a_i and b_{ij} are real. Furthermore b_{ij} are dimensionless and a_i has dimension of mass squared. Under a CP transformation of the form $\Phi_i \to \Phi_i^*$ the quadratic invariants x_1 through x_6 remain the same. But there is a change of sign for x_7, x_8 and x_9, hence the terms in V_g which are linear in these invariants break CP explicitly. Therefore the most general explicit CP breaking potential (which we have denoted by V_g) has 54 real parameters.

However for potentials which CP is not explicitly broken we are left with 30 real parameters. The potential in this case is

$$V_A(\Phi_1,\Phi_2,\Phi_3)=\sum_{i=1}^{6}a_ix_i+\frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{6}b_{ij}x_ix_j+\sum_{i=7}^{9}c_ix_i^2 \tag{13}$$

where $c_i=b_{ii}$.

To reduce the number of free parameters one can require the potential to be invariant under ($\Phi_1\to\Phi_1,\Phi_2\to\Phi_2$ and $\Phi_3\to-\Phi_3$). Now the coefficients of the terms linear in x_5 and x_6 vanish and we have left with 19 real parameters. The potential is then defined by

$$V_B(\Phi_1,\Phi_2,\Phi_3)=\sum_{i=1}^{4}a_ix_i+\frac{1}{2}\sum_{j=1}^{4}\sum_{i=1}^{4}b_{ij}x_ix_j+\sum_{i=5}^{9}c_ix_i^2 \tag{14}$$

where again $c_i=b_{ii}$.

With a $U(1)$ symmetry of the form ($\Phi_1\to\Phi_1,\Phi_2\to\Phi_2$ and $\Phi_3\to e^{i\alpha_3}\Phi_3$), (where $\alpha_3\neq 0,\pi$) we will have $b_{66}=b_{99}$ and $b_{55}=b_{88}$. Therefore the number of free parameters in this case is 17. Even though CP is not explicitly broken in this case but it is possible to break CP spontaneously.

With two $U(1)$ symmetries of the form ($\Phi_1\to\Phi_1,\Phi_2\to e^{i\alpha_2}\Phi_2$ and $\Phi_3\to e^{i\alpha_3}\Phi_3$) (where $\alpha_2\neq 0,\pi$ and $\alpha_3\neq\pm\alpha_2,0,\pi$) the potential will be

$$V_C(\Phi_1,\Phi_2,\Phi_3)=\sum_{i=1}^{3}a_ix_i+\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}b_{ij}x_ix_j+\sum_{i=4}^{9}c_ix_i^2, \tag{15}$$

but with $b_{44}=b_{77}$, $b_{66}=b_{99}$ and $b_{55}=b_{88}$. There are only 12 real parameters in V_C. In the next section we study yet another $3HDM$ potential.

3 A Special Class of Three-Higgs Doublet Models

In the last section we observed that by increasing the number of symmetry operation we can reduce the number of the parameters of the potential.

Here we impose a combination of discrete and continues symmetries on the fields. The discrete symmetries are a reflection symmetry for the field Φ_1 and an interchange symmetry for the fields Φ_2 andΦ_3 namely

$$\Phi_1\to-\Phi_1,\qquad \Phi_2\to\Phi_3,\qquad \Phi_3\to\Phi_2. \tag{16}$$

This model was discussed [13] in relation to the $\mu-\tau$ interchange symmetry. The continues symmetries of the model are comprised of three $U(1)$ for the fields. Two of these $U(1)$ symmetries namely for the fields Φ_2 and Φ_3 are related to the family symmetry of the model. The third $U(1)$ symmetry for the field Φ_1 designate the existence of one physical neutral goldstone boson. Overall this model contains only eight real parameters. In our conventions it is described by

$$V_D = a_1 x_1 + a_2(x_2 + x_3) + b_{11} x_1^2 + b_{22}(x_2^2 + x_3^2) + b_{44}(x_4^2 + x_5^2 + x_7^2 + x_8^2) \\ + b_{66}(x_6^2 + x_9^2) + b_{12} x_1 (x_2 + x_3) + b_{23} x_2 x_3. \tag{17}$$

In the model of Ref. [13] a soft breaking term is added to the potential. This term is proportional to x_6 and permits the right-handed neutrino singlet to acquire Majorana mass terms. So the model with addition of this term is

$$V_E = V_D + a_6 x_6. \tag{18}$$

To discuss the stationary points of the $3HDM$ we distinguish three types of VEV configuration

(i) Normal minima: These are defined by

$$\Phi_1 \to \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \qquad \Phi_2 \to \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \quad \text{and} \quad \Phi_3 \to \begin{pmatrix} 0 \\ v_3 \end{pmatrix}, \tag{19}$$

where v_1, v_2 and v_3 are real numbers.

(ii) Charge-breaking minima: These are defined by

$$\Phi_1 \to \begin{pmatrix} \eta_1 \\ \acute{v}_1 \end{pmatrix}, \qquad \Phi_2 \to \begin{pmatrix} 0 \\ \acute{v}_2 \end{pmatrix} \quad \text{and} \quad \Phi_3 \to \begin{pmatrix} 0 \\ \acute{v}_3 \end{pmatrix}, \tag{20}$$

where η_1, $\acute{v}_1$, $\acute{v}_2$ and $\acute{v}_3$ are real numbers.

(iii) CP-breaking minima: These have the form

$$\Phi_1 \to \begin{pmatrix} 0 \\ \tilde{v}_1 + i\delta_1 \end{pmatrix}, \qquad \Phi_2 \to \begin{pmatrix} 0 \\ \tilde{v}_2 \end{pmatrix} \quad \text{and} \quad \Phi_3 \to \begin{pmatrix} 0 \\ \tilde{v}_3 \end{pmatrix}, \tag{21}$$

where again δ_1, $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{v}_3$ are real numbers.

To discuss stationary points we must solve

$$\frac{\partial V_E}{\partial \phi_j} = 0, \quad j = 1, \ldots, 12. \tag{22}$$

Most of these equations are trivially satisfied. The non-trivial ones are

$$\frac{\partial V_E}{\partial \phi_7} = 2v_1[a_1 + 2b_{11}v_1^2 + (b_{44} + b_{12})(v_2^2 + v_3^2)] = 0, \tag{23}$$

$$\frac{\partial V_E}{\partial \phi_8} = 2a_2 v_2 + 4b_{22}v_2^3 + 2(b_{44} + b_{12})v_1^2 v_2 + 2(b_{66} + b_{23})v_2 v_3^2 + a_6 v_3 = 0, \tag{24}$$

and

$$\frac{\partial V_E}{\partial \phi_9} = 2a_2 v_3 + 4b_{22}v_3^3 + 2(b_{44} + b_{12})v_1^2 v_3 + 2(b_{66} + b_{23})v_2^2 v_3 + a_6 v_2 = 0. \tag{25}$$

In this model [13] the mass ratio of charged-leptons μ and τ is

$$\frac{m_\mu}{m_\tau} = \frac{|v_2|}{|v_3|} = \frac{1}{\kappa}. \tag{26}$$

Therefore $|v_2|$ must be much smaller than $|v_3|$. For neutral minima we assume v_1, v_2 and v_3 are real and non-negative. Furthermore if we assume the model consists of only three scalar multiplets then

$$v^2 = v_1^2 + v_2^2 + v_3^2, \tag{27}$$

where $v \simeq 246$ GeV.

We classify the solutions in several different categories:

Type I: This corresponds to

$$v_1^2 = \frac{-2b_{22}a_1 + a_2 B}{4b_{11}b_{22} - B^2}, \tag{28}$$

$$v_2^2 + v_3^2 = \frac{-2b_{11}a_2 + a_1 B}{4b_{11}b_{22} - B^2}, \quad \text{and} \quad v_2 v_3 = \frac{-a_6}{2A} \tag{29}$$

where

$$A = b_{66} + b_{23} - 2b_{22} \quad \text{and} \quad B = b_{12} + b_{44}. \tag{30}$$

Type II: In this case $v_1 = 0$ (This is the limiting case of very small v_1).

$$v_2^2 + v_3^2 = \frac{-a_2}{2b_{22}} = v^2, \quad \text{and} \quad v_2 v_3 = \frac{-a_6}{2A}. \tag{31}$$

Type III: $v_2 = v_3 = 0$ and $v_1^2 = \frac{-a_1}{2b_{11}} = v^2$ (this case corresponds to the region of parameter space where v_1 assumes values very close to v).

Now without the inclusion of the soft breaking term it is possible to obtain another group of solution. So if $a_6 = 0$ then

Type IV: Here $v_2 = 0$ and

$$v_1^2 = v_3^2 = \frac{-a_1}{2b_{11} + B} = \frac{-a_2}{2b_{22} + B} = \frac{v^2}{2}. \tag{32}$$

Type V: In this case $v_2 = 0$ but $v_1^2 \neq v_3^2$,

$$v_1^2 = \frac{a_2 B - 2a_1 b_{22}}{4b_{11}b_{22} - B^2}, \quad \text{and} \quad v_3^2 = \frac{a_1 B - 2b_{11}a_2}{4b_{11}b_{22} - B^2}. \tag{33}$$

4 The Mass Matrices of the Scalars

In our parametrization of the model we have introduced the fields φ_i, $i = 1, \ldots, 12$. The mass matrix is a 12×12 matrix. But with the choice of (1) the matrix becomes bloc-diagonal. If we assume that the couplings in scalar potential are all real then the fields corresponds to φ_{10}, φ_{11} and φ_{12} are psedoscalars. Their mass matrix is denoted by M_P^2, where the first element of this matrix is defined by

$$(M_P^2)_{1,1} = \frac{1}{2}\frac{\partial^2 V_E}{\partial\varphi_{10}\partial\varphi_{10}}. \tag{34}$$

and it is

$$M_P^2 = \begin{pmatrix} a_1 + 2b_{11}v_1^2 + B(v_2^2+v_3^2) & 0 & 0 \\ 0 & a_2 + 2b_{22}v_2^2 + Bv_1^2 & 0.5a_6 \\ 0 & 0.5a_6 & a_2 + 2b_{22}v_3^2 + Bv_1^2 \end{pmatrix} + k\begin{pmatrix} 0 & 0 & 0 \\ 0 & v_3^2 & 0 \\ 0 & 0 & v_2^2 \end{pmatrix}, \tag{35}$$

where $k = b_{23} + b_{66}$.

The fields that corresponds to φ_7, φ_8 and φ_9 are scalars. Their mass matrix is denoted by M_S^2, where the first element of this matrix is defined by

$$(M_S^2)_{1,1} = \frac{1}{2}\frac{\partial^2 V_E}{\partial\varphi_7\partial\varphi_7} \tag{36}$$

and it is

$$M_S^2 = \begin{pmatrix} a_1 + 6b_{11}v_1^2 + B(v_2^2+v_3^2) & 2Bv_1v_2 & 2Bv_1v_3 \\ 2Bv_1v_2 & a_2 + 6b_{22}v_2^2 + Bv_1^2 & 0.5a_6 \\ 2Bv_1v_3 & 0.5a_6 & a_2 + 6b_{22}v_3^2 + Bv_1^2 \end{pmatrix} + k\begin{pmatrix} 0 & 0 & 0 \\ 0 & v_3^2 & 2v_2v_3 \\ 0 & 2v_2v_3 & v_2^2 \end{pmatrix}. \tag{37}$$

And there is no mixing between the scalars and pseudoscalars.

The remaining six fields will corresponds to the charged Higgs. If We introduce

$$\zeta_1 = \phi_1 + i\phi_2, \qquad \zeta_2 = \phi_3 + i\phi_4 \quad \text{and} \quad \zeta_3 = \phi_5 + i\phi_6 \tag{38}$$

and

$$\zeta_1^* = \phi_1 - i\phi_2, \qquad \zeta_2^* = \phi_3 - i\phi_4 \quad \text{and} \quad \zeta_3^3 = \phi_5 - i\phi_6 \tag{39}$$

then the mass matrix of the charged Higgs is defined by

$$(M_C^2)_{\alpha\beta} = \frac{\partial^2 V_E}{\partial\zeta_\beta\partial\zeta_\alpha^*} \quad \text{with } \alpha, \beta = 1, 2, 3. \tag{40}$$

By using the invariants defined in section two and after some algebra we obtain

$$M_C^2 = \begin{pmatrix} a_1 + 2b_{11}v_1^2 + b_{12}(v_2^2+v_3^2) & b_{44}v_1v_2 & b_{44}v_1v_3 \\ b_{44}v_1v_2 & a_2 + 2b_{22}v_2^2 + b_{12}v_1^2 & 0.5a_6 + b_{66}v_2v_3 \\ b_{44}v_1v_3 & 0.5a_6 + b_{66}v_2v_3 & a_2 + 2b_{22}v_3^2 + b_{12}v_1^2 \end{pmatrix} + b_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & v_3^2 & 0 \\ 0 & 0 & v_2^2 \end{pmatrix}. \tag{41}$$

In order to have a true minima the eigenvalues of the mass matrix must be non-negative. We assume that the VEVs of the scalars are strictly positive then for the case of the pseudo-scalar mass matrix M_P^2 the eigenvalues are

$$\lambda_1 = 0, \qquad \lambda_2 = 0 \quad \text{and} \quad \lambda_3 = A(v_2^2 + v_3^2). \tag{42}$$

For the charged scalar M_C^2 the eigenvalues are

$$\begin{aligned} &\lambda_4 = 0, \qquad \lambda_5 = -b_{44}v^2 \quad \text{and} \\ &\lambda_6 = -b_{44}v_1^2 + (b_{23} - 2b_{22})(v_2^2 + v_3^2). \end{aligned} \tag{43}$$

From the positivity of these mass matrices we must have

$$A > 0, \qquad b_{44} < 0 \quad \text{and} \quad \lambda_6 > 0. \tag{44}$$

In general it is not possible to obtain closed analytical expressions for the eigenvalues of the mass matrix M_S^2. However they satisfy the cubic equation

$$\lambda^3 + \gamma_1\lambda^2 + \gamma_2\lambda + \gamma_3 = 0, \tag{45}$$

where

$$\begin{aligned} &\gamma_1 = -4b_{11}v_1^2 - (4b_{22} + A)(v_2^2 + v_3^2), \\ &\gamma_3 = -4Av_1^2(v_3^2 - v_2^2)^2(4b_{11}b_{22} - B^2) \end{aligned} \tag{46}$$

and

$$\gamma_2 = 4\big\{[b_{11}(4b_{22} + A) - B^2]v_1^2(v_2^2 + v_3^2) + b_{22}A(v_3^2 - v_2^2)^2\big\}. \tag{47}$$

But from the properties of a cubic equation we know that

$$\begin{aligned} &\lambda_7 + \lambda_8 + \lambda_9 = -\gamma_1, \qquad \lambda_7\lambda_8\lambda_9 = -\gamma_3 \quad \text{and} \\ &\lambda_7\lambda_8 + \lambda_7\lambda_9 + \lambda_8\lambda_9 = \gamma_2. \end{aligned} \tag{48}$$

But b_{11} and b_{22} must be positive in order for the potential to have a minimum. So if we require $4b_{11}b_{22} > B^2$ then all the eigenvalues will be positive. We must also include the condition of having real roots that for our case becomes

$$27(2\gamma_2^3 + 9\gamma_3^2 + 2\gamma_3\gamma_1^3) < 9\gamma_1^2\gamma_2^2 + 8\gamma_1^4\gamma_2 + 162\gamma_1\gamma_2\gamma_3. \tag{49}$$

5 The Structure of the Normal Minima

From our solution of Type I we find $a_6 < 0$ and

$$a_2B > 2a_1b_{22}, \qquad a_1B > 2a_2b_{11}, \tag{50}$$

And the value of the potential at this minimum is

$$\begin{aligned} (V_E)_I = {} & -b_{11}v_1^4 - b_{22}(v_2^4 + v_3^4) \\ & - (b_{44} + b_{12})v_1^2(v_2^2 + v_3^2) - (b_{66} + b_{23})v_2^2v_3^2. \end{aligned} \tag{51}$$

And the value of potential for Type II solution is

$$(V_E)_{II} = -[b_{22}(1+\kappa^4) + (b_{66}+b_{23})\kappa^2]\frac{v^4}{(1+\kappa^2)^2} \tag{52}$$

and for Type III $(V_E)_{III} = -b_{11}v^4$ and for Type IV it is

$$(V_E)_{IV} = -[b_{11}+b_{22}+b_{44}+b_{12}]\frac{v^4}{4}. \tag{53}$$

And finally for Type V we have

$$(V_E)_V = -b_{11}v_1^4 - b_{22}(v^2-v_1^2)^2 - (b_{44}+b_{12})v_1^2(v^2-v_1^2). \tag{54}$$

We notice that the values of the potential at neutral minima related to the types (II, III, IV) are fixed. But as we see from (54) this value for Type V is a function of v_1, to find the deepest minimum in this category

$$\frac{\partial(V_E)_V}{\partial v_1} = 0 \quad \Rightarrow \quad v_1^2 = \frac{(2b_{22}-B)v^2}{2(B-b_{11}-b_{22})} = \theta v^2. \tag{55}$$

And the value of the potential is

$$(V_E)_{V\,deep} = -b_{11}v^4\left[\theta^2 - \frac{b_{22}}{b_{11}}(1-\theta)^2 - B\theta(1-\theta)\right]. \tag{56}$$

6 CP Breaking Stationary Points

Even though in this model CP is not explicitly broken. But if choose the vacua as in (21), then CP symmetry is broken spontaneously.

To discuss CP stationary points we must solve

$$\frac{\partial V_E}{\partial \phi_j} = 0, \quad j = 1, \ldots, 12. \tag{57}$$

Most of these equations are trivially satisfied. The non-trivial ones are

$$\frac{\partial V_E}{\partial \phi_7} = 2\tilde{v}_1[a_1 + 2b_{11}(\tilde{v}_1^2+\delta^2) + (b_{44}+b_{12})(\tilde{v}_2^2+\tilde{v}_3^2)] = 0, \tag{58}$$

$$\begin{aligned}\frac{\partial V_E}{\partial \phi_8} &= 2a_2\tilde{v}_2 + 4b_{22}\tilde{v}_2^3 + 2(b_{44}+b_{12})(\tilde{v}_1^2+\delta^2)\tilde{v}_2 \\ &\quad + 2(b_{66}+b_{23})\tilde{v}_2\tilde{v}_3^2 + a_6\tilde{v}_3 \\ &= 0,\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{\partial V_E}{\partial \phi_9} &= 2a_2\tilde{v}_3 + 4b_{22}\tilde{v}_3^3 + 2(b_{44}+b_{12})(\tilde{v}_1^2+\delta^2)\tilde{v}_3 \\ &\quad + 2(b_{66}+b_{23})\tilde{v}_2^2\tilde{v}_3 + a_6\tilde{v}_2 \\ &= 0,\end{aligned} \tag{60}$$

and

$$\frac{\partial V_E}{\partial \phi_{10}} = 2\delta[a_1 + 2b_{11}(\tilde{v}_1^2 + \delta^2) + (b_{44} + b_{12})(\tilde{v}_2^2 + \tilde{v}_3^2)] = 0. \tag{61}$$

But if we assume $\tilde{v}_1 \neq 0$ and $\delta \neq 0$. Then (58) and (61) are identical and we have only three independent expressions for the CP breaking stationary points.

The charged Higgs mass matrix in this case is

$$M_C^2 = \begin{pmatrix} a_1 + 2b_{11}\Delta + b_{12}(\tilde{v}_2^2 + \tilde{v}_3^2) & b_{44}(\tilde{v}_1 + i\delta)\tilde{v}_2 & b_{44}(\tilde{v}_1 + i\delta)\tilde{v}_3 \\ b_{44}(\tilde{v}_1 - i\delta)\tilde{v}_2 & a_2 + 2b_{22}\tilde{v}_2^2 + b_{12}\Delta + b_{23}\tilde{v}_3^2 & 0.5a_6 + b_{66}\tilde{v}_2\tilde{v}_3 \\ b_{44}(\tilde{v}_1 - i\delta)\tilde{v}_3 & 0.5a_6 + b_{66}\tilde{v}_2\tilde{v}_3 & a_2 + 2b_{22}\tilde{v}_3^2 + b_{12}\Delta + b_{23}\tilde{v}_2^2 \end{pmatrix} \tag{62}$$

where $\Delta = \tilde{v}_1^2 + \delta^2$. By using the CP breaking stationary points conditions we obtain the eigenvalues of this matrix which corresponds to the masses of the charged Higgs. They are

$$\begin{aligned} &\tilde{\lambda}_4 = 0, \qquad \tilde{\lambda}_5 = -b_{44}v^2 \quad \text{and} \\ &\tilde{\lambda}_6 = -b_{44}\Delta + (b_{23} - 2b_{22})(\tilde{v}_2^2 + \tilde{v}_3^2). \end{aligned} \tag{63}$$

For the normal vacua as the VEVs are real there is no scalar pseudoscalar mixing. However CP breaking vacua causes a mixing between the scalars and psedoscalars. In the appendix we discuss this mass matrix. Three of the eigenvalues are

$$\tilde{\lambda}_1 = 0, \qquad \tilde{\lambda}_2 = 0 \quad \text{and} \quad \tilde{\lambda}_3 = A(\tilde{v}_2^2 + \tilde{v}_3^2). \tag{64}$$

In Appendix A we show that if the conditions $A > 0$ and $4b_{11}b_{22} > B^2$ holds then the other three eigenvalues are positive and hence the stationary points in this case is a true minima.

In Ref. [12] it is shown that if both normal and CP breaking vacua exist then the normal minima lies below a CP breaking minima. So the CP breaking vacua is an unstable one.

7 Conclusions

We have tried to initiate a systematic and quantitative study of three Higgs doublet models. And we studied the structure of the neutral minima within a specific $3HDM$. At charged-breaking stationary point one of the charged fields φ has a non-zero VEV. It will be of interest to study the structure of charged breaking vacua of this model and compare the results with that of neutral minima.

The three Higgs doublet model expressed by (15) has only 12 real parameters, it will be of interest to study the vacuum structure of this case as well.

Appendix A: The scalar pseudoscalar mixing

In this appendix we study a 6×6 mass matrix defined by

$$(M^2)_{i,j} = \frac{1}{2}\frac{\partial^2 V_E}{\partial\varphi_{(6+i)}\partial\varphi_{(6+j)}}, \qquad i, j = 1, 2, 3, 4, 5, 6. \tag{65}$$

The elements of this mass matrix are

$$
\begin{aligned}
&M_{11}^2 = a_1 + 6b_{11}\Delta + B(\tilde{v}_2^2 + \tilde{v}_2^3).\\
&M_{12}^2 = 2B(\tilde{v}_1 + i\delta)\tilde{v}_2, \qquad M_{13}^2 = 2B(\tilde{v}_1 + i\delta)\tilde{v}_3,\\
&M_{14}^2 = 4b_{11}\delta\tilde{v}_1, \qquad M_{15}^2 = M_{16}^2 = 0.\\
&M_{22}^2 = a_2 + 6b_{22}\tilde{v}_2^2 + B\Delta + k\tilde{v}_3^2.\\
&M_{23}^2 = 0.5a_6 + 2k\tilde{v}_2\tilde{v}_3, \qquad M_{24}^2 = 2B\tilde{v}_2\delta,\\
&M_{25}^2 = M_{26}^2 = 0.\\
&M_{33}^2 = a_2 + 6b_{22}\tilde{v}_3^2 + B\Delta + k\tilde{v}_2^2.\\
&M_{34}^2 = 2B\tilde{v}_3\delta, \qquad M_{35}^2 = M_{36}^2 = 0.\\
&M_{44}^2 = a_1 + 2b_{11}\Delta + B(\tilde{v}_2^2 + \tilde{v}_3^2) + 4b_{11}\delta^2.\\
&M_{45}^2 = M_{46}^2 = 0.\\
&M_{55}^2 = a_2 + 2b_{22}\tilde{v}_2^2 + B\Delta + k\tilde{v}_3^2, \qquad M_{56}^2 = 0.5a_6.\\
&M_{66}^2 = a_2 + 2b_{22}\tilde{v}_3^2 + B\Delta + k\tilde{v}_2^2.
\end{aligned}
$$

Therefore this 6×6 matrix is block diagonal and decomposes to a 4×4 square matrix and another 2×2 square matrix.

Next we calculate the eigenvalues of this 2×2 square matrix. The result is $\tilde{\lambda}_2 = 0$ and $\tilde{\lambda}_3 = A(\tilde{v}_2^2 + \tilde{v}_3^2)$.

And finally we consider the eigenvalues of the 4×4 square matrix. It turns out that one of the eigenvalues $\tilde{\lambda}_1 = 0$, and the three remaining eigenvalues satisfy the cubic equation

$$\tilde{\lambda}^3 + \tilde{\gamma}_1\tilde{\lambda}^2 + \tilde{\gamma}_2\tilde{\lambda} + \tilde{\gamma}_3 = 0, \tag{66}$$

where

$$
\begin{aligned}
\tilde{\gamma}_1 &= -4b_{11}\Delta - (4b_{22} + A)(\tilde{v}_2^2 + \tilde{v}_3^2),\\
\tilde{\gamma}_3 &= -4A\Delta(\tilde{v}_3^2 - \tilde{v}_2^2)^2(4b_{11}b_{22} - B^2)
\end{aligned}
\tag{67}
$$

and

$$\tilde{\gamma}_2 = 4\{[b_{11}(4b_{22} + A) - B^2]\Delta(\tilde{v}_2^2 + \tilde{v}_3^2) + b_{22}A(\tilde{v}_3^2 - \tilde{v}_2^2)^2\}. \tag{68}$$

Again the argument of section four applies here and with conditions $A > 0$ and $4b_{11}b_{22} > B^2$ these eigenvalues will be positive as well. And hence the CP breaking stationary points are indeed the true minima. In deriving these results we have used (58)–(61).

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